Involutivity of the Hamilton-Cartan equations of a second-order Lagrangian admitting a first-order Hamiltonian formalism

Jaime Muñoz-Masqué, CSIC , Spain
M. Eugenia Rosado, ETSAM, UPM, Spain

July 2012

Introduction

$p\colon E \to N$ fibred manifold, v volume form on N
$\Lambda = Lv$, $L \in C^\infty(J^2E)$, second-order Lagrangian density on $p\colon E \to N$.

Hamilton-Cartan equations of Λ

Generally there is no equivalence between Euler-Lagrange equations and Hamilton-Cartan equations because the Poincaré-Cartan form of a Lagrangian on J^2E is generally defined on J^3E.

Problem: To classify second-order Lagrangians whose Poincaré-Cartan form is projectable onto J^2E or even to J^1E.

In this case, we study the involutivity of the Hamilton-Cartan equations.

N oriented connected smooth manifold, $n = \dim N$.
$p\colon E \to N$ fibred manifold over N.
Let $p^k\colon J^k E \to N$ be the k-jet bundle of p, with natural projections

$$p^k_h\colon J^k E \longrightarrow J^h E, \quad k \geq h.$$

Coordinates induced by (x^i, y^α), $1 \leq i \leq n$, $1 \leq \alpha \leq m = \dim E - \dim N$ on $J^2 E$:

$$(x^i, y^\alpha, y^\alpha_i, y^\alpha_{(ij)}),$$

where

$$y^\alpha_i(j^2_x s) = \frac{\partial(y^\alpha \circ s)}{\partial x^i}(x), \quad y^\alpha_{(ij)}(j^2_x s) = \frac{\partial^2(y^\alpha \circ s)}{\partial x^i \partial x^j}(x).$$

The Poincaré-Cartan form attached to Λ is defined to be the ordinary n-form on J^3E given by

$$\Theta_\Lambda = (p_2^3)^*\theta^2 \wedge \omega_\Lambda + \Lambda,$$

where θ^2 is the second-order structure form on J^2E
and ω_Λ is the Legendre form of Λ,

$$\begin{cases} \theta^2 = \left(dy^\alpha - y_k^\alpha dx^k\right) \otimes \dfrac{\partial}{\partial y^\alpha} + \left(dy_h^\alpha - y_{(hk)}^\alpha dx^k\right) \otimes \dfrac{\partial}{\partial y_h^\alpha}, \\ \omega_\Lambda = (-1)^{i-1} L_\alpha^{i0} v_i \otimes dy^\alpha + (-1)^{i-1} L_\alpha^{ij} v_i \otimes dy_j^\alpha, \end{cases}$$

$v_i = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$ and the exterior product is taken with respect to the pairing induced by duality, $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$.

In coordinates Θ_Λ is given by

$$\Theta_\Lambda = (-1)^{i-1}\left(L^{i0}_\alpha dy^\alpha + L^{ih}_\alpha dy^\alpha_h\right)\wedge v_i + \left(L - y^\alpha_i L^{i0}_\alpha - y^\alpha_{(hi)} L^{ih}_\alpha\right) v$$

where

$$\begin{cases} L^{ij}_\alpha = \frac{1}{2-\delta_{ij}}\dfrac{\partial L}{\partial y^\alpha_{(ij)}}, \\ L^{i0}_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \frac{1}{2-\delta_{ij}} D_j\left(\frac{\partial L}{\partial y^\alpha_{(ij)}}\right). \end{cases}$$

D_j denoting the "total derivative" with respect to x^j, i.e.,

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|I|=0}^{\infty}\sum_{\alpha=1}^{m} y^\alpha_{I+(j)}\frac{\partial}{\partial y^\alpha_I}.$$

It is known that the Poincaré-Cartan form of a second-order Lagrangian is projectable onto J^2E if and only if the following system of PDEs holds:

$$\tfrac{1}{2-\delta_{ib}}\frac{\partial^2 L}{\partial y^\beta_{ac}\partial y^\alpha_{ib}}+\tfrac{1}{2-\delta_{ia}}\frac{\partial^2 L}{\partial y^\beta_{bc}\partial y^\alpha_{ia}}+\tfrac{1}{2-\delta_{ic}}\frac{\partial^2 L}{\partial y^\beta_{ab}\partial y^\alpha_{ic}}=0,$$

for all indices $1\leq a\leq b\leq c\leq n$, $\alpha,\beta=1,\ldots,m$.

R. Durán Díaz, J. Muñoz Masqué, *Second-order Lagrangians admiting a second-order Hamilton-Cartan formalism*, J. Phys. A: Math. Gen. **33** (2000), 6003–6016.

Projecting onto J^1E

$p_1^2\colon J^2E \longrightarrow J^1E$ admits an affine bundle structure modelled over $W^2 = (p^1)^*S^2T^*N \otimes (p_0^1)^*V(p) \longrightarrow J^1E$.

Theorem

Θ_Λ projects onto J^1E if and only if

$$L = L_\alpha^{ij} y_{(ij)}^\alpha + L_0, \quad L_\alpha^{ji} = L_\alpha^{ij} \in C^\infty(J^1E), L_0 \in C^\infty(J^1E),$$

$$0 = 2\frac{\partial L_\beta^{hi}}{\partial y_a^\alpha} - \frac{\partial L_\alpha^{ai}}{\partial y_h^\beta} - \frac{\partial L_\alpha^{ah}}{\partial y_i^\beta} \Leftrightarrow \boxed{\frac{\partial L_\beta^{ih}}{\partial y_a^\alpha} = \frac{\partial L_\alpha^{ia}}{\partial y_h^\beta},}$$

$a, h, i = 1, \ldots, n$, and $\alpha, \beta = 1, \ldots, m$. Hence, there exists $L^i \in C^\infty(J^1E)$ such that locally,

$$\boxed{L_\alpha^{ih} = \frac{\partial L^i}{\partial y_h^\alpha}} \quad \textit{and} \quad \frac{\partial L^h}{\partial y_i^\alpha} = \frac{\partial L^i}{\partial y_h^\alpha}.$$

Hamiltonian formalism (1)

For $\Lambda = Lv$, $L \in C^\infty(J^2E)$ we have:

$$d\Theta_\Lambda = \mathcal{E}_\alpha(L)\theta^\alpha \wedge v + (-1)^i \eta_2^i(L) \wedge v_i,$$

where $\eta_2^i(L)$ is the 2-contact 2-form given by,

$$\eta_2^i(L) = \frac{\partial L_\alpha^{i0}}{\partial y^\beta}\theta^\alpha \wedge \theta^\beta + \left(\frac{\partial L_\alpha^{i0}}{\partial y_j^\beta} - \frac{\partial L_\beta^{ij}}{\partial y^\alpha}\right)\theta^\alpha \wedge \theta_j^\beta + \sum_{j\leq k}\frac{\partial L_\alpha^{i0}}{\partial y_{(jk)}^\beta}\theta^\alpha \wedge \theta_{(jk)}^\beta$$
$$+ \sum_{i\leq k\leq l}\frac{\partial L_\alpha^{i0}}{\partial y_{(jkl)}^\beta}\theta^\alpha \wedge \theta_{(jkl)}^\beta + \frac{\partial L_\alpha^{ij}}{\partial y_k^\beta}\theta_j^\alpha \wedge \theta_k^\beta + \sum_{k\leq l}\frac{\partial L_\alpha^{ij}}{\partial y_{(kl)}^\beta}\theta_j^\alpha \wedge \theta_{(kl)}^\beta.$$

Hence the Hamilton-Cartan equations also characterize critical sections for Λ; i.e., for every p^3-vertical vector field X on J^3E,

$$s \text{ is an extremal for } \Lambda \Longleftrightarrow (j^3s)^*(i_X d\Theta_\Lambda) = 0.$$

Hamiltonian formalism (2)

If $L \in C^\infty(J^2E)$ whose Poincaré-Cartan form projects onto J^1E, then letting

$$\begin{cases} p^i_\alpha = L^{i0}_\alpha - \dfrac{\partial L^i}{\partial y^\alpha}, & 1 \leq \alpha \leq m,\ 1 \leq i \leq n, \\ H = L_0 - y^\alpha_i L^{i0}_\alpha - \dfrac{\partial L^i}{\partial x^i}, & \text{Hamiltonian function of } L \end{cases}$$

one obtains,

$$d\Theta_\Lambda = (-1)^{i-1} dp^i_\alpha \wedge dy^\alpha \wedge v_i + dH \wedge v.$$

Furthermore, if $d_{10}(p^i_\alpha)\colon V(p^1_0) \to \mathbb{R}$ are linearly independent, then $s\colon N \to E$ is an extremal for Λ if and only if it satisfies:

$$\begin{cases} 0 = \dfrac{\partial(p^i_\alpha \circ j^1 s)}{\partial x^i} - \dfrac{\partial H}{\partial y^\alpha} \circ j^1 s, & 1 \leq \alpha \leq m, \\ 0 = \dfrac{\partial(y^\alpha \circ s)}{\partial x^i} + \dfrac{\partial H}{\partial p^i_\alpha} \circ j^1 s, & 1 \leq \alpha \leq m,\ 1 \leq i \leq n. \end{cases}$$

Regularity

Theorem

If $\Lambda = Lv$ is a second-order Lagrangian on E such that,

- *its Poincaré-Cartan form Θ_Λ projects onto J^1E,*
- *the linear forms $d_{10}(p_\alpha^i) \colon V(p_0^1) \to \mathbb{R}$, $1 \leq \alpha \leq m$, $1 \leq i \leq n$, are linearly independent,*

then every solution to its H-C equations, is holonomic.

Idea If the linear forms $d_{10}(p_\alpha^i)$ are linearly independent hence

$$\left(\frac{\partial p_\alpha^i}{\partial y_h^\beta}\right) \text{ is non-singular}$$

and $(s^1)^*\left(i_{\partial/\partial y_h^\alpha} d\Theta_\Lambda\right) = 0$ implies $s^1 = j^1 s$ with $s = p_0^1 \circ s^1$.

Goal

Let $L \in C^\infty(J^2E)$ whose Poincaré-Cartan form projects onto J^1E.

Let $p' \colon E' \to N$ be the bundle defined as follows: $p' = p^1$, $E' = J^1E$.

Let $R^1_L \subset J^1E'$ be the first-order differential system defined by the solutions to the Hamilton-Cartan equations of $\Lambda = Lv$.

- We study the involutivity of $R^1_L \subset J^1E'$.
- According to the regularity, for every (local) solution s^1 to R^1_L one has $s^1 = j^1 s$, where the section of $p \colon E \to N$ defined by $s = p^1_0 \circ s^1$ is called the zero-order section attached to s^1.

We consider the morphism of fibred manifolds over E'

$$\begin{aligned}\varphi_L \colon J^1E' &\to B_N = (p')^* \oplus^{m+mn} (\wedge^n T^*N), \\ j^1_x s^1 &\longmapsto \left(s^1(x);\ \left(\varphi^\alpha_L(j^1_x s^1)v\right)_{1\le\alpha\le m},\ \left(\left(\varphi^i_\alpha\right)_L (j^1_x s^1)v\right)^{1\le i\le n}_{1\le\alpha\le m}\right)\end{aligned}$$

defined by,

$$\begin{cases} \varphi^\alpha_L(j^1_x s^1)v = (s^1)^* \left(i_{\partial/\partial y^\alpha} d\Theta_\Lambda\right)(x), \\ \left(\varphi^i_\alpha\right)_L (j^1_x s^1)v = (s^1)^* \left(i_{\partial/\partial p^i_\alpha} d\Theta_\Lambda\right)(x). \end{cases}$$

One has $R^1_L = \ker \varphi_L$, where the morphism $\varphi_L \colon J^1E' \longrightarrow B_N$ is given in local coordinates by

$$\left\{ \begin{array}{l} \varphi^\alpha_L = -p^i_{\alpha,i} + H_{y^\alpha}, \\ \left(\varphi^i_\alpha\right)_L = y^\alpha_{,i} + H_{p^i_\alpha}. \end{array} \right.$$

where $(x^i, y^\alpha, p^i_\alpha)$ is a fibred coordinate system for $p' \colon E' \to N$, and $(x^i, y^\alpha, p^i_\alpha; y^\alpha_{,j}, p^i_{\alpha,j})$ be the induced coordinate system on J^1E'.

The mapping $\varphi_L\colon J^1E' \to B_N$ is quasi-linear as there exists a vector-bundle morphism, the symbol of φ_L,

$$\sigma_L\colon (p')^* T^* N \otimes V(p') \to B_N,$$

such that,

$$\varphi_L(j^1_x\bar{s}^1) = \sigma_L(\chi_1) + \varphi_L(j^1_x s^1),$$

$\forall \chi_1 \in T^*_x N \otimes V_{e'}(p')$, $\forall j^1_x s^1 \in J^1E'$, with $j^1_x s^1 + \chi_1 = j^1_x \bar{s}^1$, $\bar{s}^1(x) = s^1(x) = e'$.
If $\chi_1 = (dx^j)_x \otimes \{t^\alpha_j(\chi_1)(\partial/\partial y^\alpha)_{e'} + t^\alpha_{ij}(\chi_1)(\partial/\partial p^i_\alpha)_{e'}\}$, then

$$\begin{cases} y^\alpha_{,j}(j^1_x\bar{s}^1) = t^\alpha_j(\chi_1) + y^\alpha_{,j}(j^1_x s^1), \\ p^i_{\alpha,j}(j^1_x\bar{s}^1) = t^\alpha_{ij}(\chi_1) + p^i_{\alpha,j}(j^1_x s^1). \end{cases}$$

As $H_{y^\alpha}, H_{p^i_\alpha} \in C^\infty(E')$ and $\bar{s}^1(x) = s^1(x) = e'$, we have

$$H_{y^\alpha}(j^1_x\bar{s}^1) = H_{y^\alpha}(j^1_x s^1), \quad H_{p^i_\alpha}(j^1_x\bar{s}^1) = H_{p^i_\alpha}(j^1_x s^1).$$

Therefore

$$\begin{cases} \varphi^\alpha_L = -p^i_{\alpha,i} + H_{y^\alpha}, \\ \left(\varphi^i_\alpha\right)_L = y^\alpha_{,i} + H_{p^i_\alpha}, \end{cases} \Longrightarrow \begin{cases} u^\alpha \circ \sigma_L = -\sum_{h=1}^n t^\alpha_{hh}, \\ u^i_\alpha \circ \sigma_L = t^\alpha_i, \end{cases}$$

(u^α, u^i_α) being the standard coordinates induced by the volume form in $B_N = \oplus^{m+mn}(\wedge^n T^*N)$. Hence

$$\begin{aligned} \mathfrak{g}_1 &= \ker \sigma_L \\ &= \left\{\chi_1 \in T^*_x N \otimes V_{e'}(p') : \textstyle\sum_{h=1}^n t^\alpha_{hh}(\chi_1) = 0, t^\alpha_i(\chi_1) = 0\right\}. \end{aligned}$$

R. Bryant, S. S. Chern, R. Gardner, H. Goldschmidt, and P. Griffiths, *Exterior differential systems*, Math. Sci. Res. Inst. Publ., Volume 18, Springer-Verlag, Berlin and New York, 1991.

The PDEs system $R^1_L \subset J^1(E')$ is formally integrable if $\forall k \geq 0$,

$$p^{1+k}_k \colon R^{1+k}_L \longrightarrow R^k_L$$

are surjective, $R^{1+k}_L \subset J^{1+k}(E')$ being the k-prolongation of R^1_L.

Theorem

If the following conditions are satisfied:

1. $\mathfrak{g}_2 = \ker\,(\sigma_L)_1$ *is a vector bundle over* R^1_L*, where* $(\sigma_L)_1$ *is the first-prolongation of the symbol of* R^1_L*,*
2. $p^1_0 \colon R^1_L \longrightarrow E'$ *is surjective,*
3. *there exists a quasi-regular basis for* T_x *for* $\mathfrak{g}_1 = \ker\sigma_L$ *at* $u \in P$*,*

then, the PDEs system R^1_L *is formally integrable.*

The first prolongation of φ_L is defined by,

$$\operatorname{prol}_1(\varphi_L)\colon J^2(E') \to J^1 B_N,$$
$$\operatorname{prol}_1(\varphi_L)(j^2_x s^1) = j^1_x(\varphi \circ j^1 s^1).$$

In coordinates one has,

$$\begin{cases} u^\alpha \circ \operatorname{prol}_1(\varphi_L) = \varphi^\alpha_L = -p^i_{\alpha,i} + H_{y^\alpha}, \\ u^i_\alpha \circ \operatorname{prol}_1(\varphi_L) = \left(\varphi^i_\alpha\right)_L = y^\alpha_{,i} + H_{p^i_\alpha}, \\ u^\alpha_r \circ \operatorname{prol}_1(\varphi_L) = -p^i_{\alpha,ir} + H_{x^r y^\alpha} + y^\beta_{,r} H_{y^\alpha y^\beta} + p^j_{\beta,r} H_{y^\alpha p^j_\beta}, \\ u^i_{\alpha,r} \circ \operatorname{prol}_1(\varphi_L) = y^\alpha_{,ir} + H_{x^r p^i_\alpha} + y^\beta_{,r} H_{y^\beta p^i_\alpha} + p^j_{\beta,r} H_{p^i_\alpha p^j_\beta}, \end{cases}$$

$(u^\alpha, u^i_\alpha, u^\alpha_r, u^i_{\alpha,r})$ being the induced coordinates on $J^1 B_N$.

As $\varphi_L \colon J^1(E') \to B_N$ is quasi-linear: $\sigma_k(\varphi_L) = \sigma\left(\mathrm{prol}_k(\varphi_L)\right)$, $\forall k \geq 0$.

Hence $\forall \chi_2 \in S^2 T_x^* N \otimes V_{e'}(p')$, $\forall j_x^2 s^1 \in J^2 E'$, we have:

$$\sigma_1(\varphi_L) \colon (p')^* S^2 T^* N \otimes V(p') \to (p')^* T^* N \otimes B_N,$$
$$\sigma_1(\varphi_L)(\chi_2) = \mathrm{prol}_1(\varphi_L)(\chi_2 + j_x^2 s^1) - \mathrm{prol}_1(\varphi_L)(j_x^2 s^1).$$

If $\chi_2 = \sum_{j \leq k} (dx^j)_x \odot (dx^k)_x \otimes \left\{ t_{jk}^{\alpha}(\chi_2)(\partial/\partial y^{\alpha})_{e'} + t_{ijk}^{\alpha}(\chi_2)(\partial/\partial p_{\alpha}^i)_{e'} \right\}$,

then

$$\begin{cases} u_r^{\beta} \circ \sigma_1(\varphi_L) = -\sum_{a=1}^{n} (1 + \delta_{ar}) t_{aar}^{\beta}, \\ u_{\beta,r}^{i} \circ \sigma_1(\varphi_L) = (1 + \delta_{ir}) t_{ir}^{\beta}. \end{cases}$$

Let $(X_1, \ldots, X_n)$ be a basis of $T_x N$, with dual basis $(w^1, \ldots, w^n)$ and let $S^k T^*_{x,\{X_1,\ldots,X_j\}}$ be the subspace of $S^k T^*_x$ generated by the symmetric products $w^{i_1} \odot \cdots \odot w^{i_k}$, with $j+1 \leq i_1 \leq \cdots \leq i_k \leq n$. For every $e' \in E'$ with $p'(e') = x$, we set

$$\mathfrak{g}_{k,e',\{X_1,\ldots,X_j\}} = \mathfrak{g}_{k,e'} \cap \left(S^k T^*_{x,\{X_1,\ldots,X_j\}} \otimes V_{e'}(p')\right), \quad k = 1, 2.$$

The basis $(X_1, \ldots, X_n)$ is quasi-regular for $\mathfrak{g}_1$ at e' if

$$\dim \mathfrak{g}_{2,e'} = \dim \mathfrak{g}_{1,e'} + \sum_{j=1}^{n-1} \dim \mathfrak{g}_{1,e',\{X_1,\ldots,X_j\}}.$$

As

$$\mathfrak{g}_1 = \left\{ \chi_1 \in T_x^* N \otimes V_{e'}(p') : \sum_{h=1}^n t_{hh}^\alpha(\chi_1) = 0, t_i^\alpha(\chi_1) = 0 \right\},$$

$$\mathfrak{g}_2 = \left\{ \chi_2 \in S^2 T_x^* N \otimes V_{e'}(p') : \sum_{a=1}^n (1 + \delta_{ar}) t_{aar}^\beta(\chi_2) = 0, t_{ir}^\beta(\chi_2) = 0 \right\},$$

we conclude

$$\begin{aligned} \dim \mathfrak{g}_{1,e'} &= m(n^2 - 1), \\ \dim \mathfrak{g}_{2,e'} &= \tfrac{1}{2} mn(n^2 + n - 2), \\ \dim \mathfrak{g}_{1,e',\{X_1,\ldots,X_j\}} &= m(n^2 - jn - 1). \end{aligned}$$

Hence

$$\sum_{j=1}^{n-1} \dim \mathfrak{g}_{1,e',(X_1,\ldots,X_j)} = \tfrac{1}{2} m(n-1)(n^2 - 2) = \dim \mathfrak{g}_{2,e'} - \dim \mathfrak{g}_{1,e'},$$

thus proving that R_L^1 is involutive .

Involutivity

Theorem

If $\Lambda = L v$ is a second-order density on $p\colon E \to N$ whose P-C form projects onto J^1E and satisfies the regularity condition, then R^1_L is involutive.
If both $p\colon E \to N$ and Λ are of class C^ω, then given $\xi \in (R^1_L)_{x_0}$, there exists a C^ω section s of p defined on an open neighbourhood U of x_0 in N such that, i) $j^1_{x_0}(j^1 s) = \xi$, and ii) $j^1_x(j^1 s) \in R^1_L$, $\forall x \in U$.

Analytically, given scalars λ^α, λ^i_α, λ^α_j, $\lambda^i_{\alpha,j}$ such that,

$$
\begin{aligned}
0 &= \lambda^i_{\alpha,i} - H_{y^\alpha}(e'_0),\\
0 &= \lambda^\alpha_i + H_{p^i_\alpha}(e'_0),
\end{aligned}
$$

where $e'_0 \in E'_{x_0}$ with coordinates $y^\alpha(e') = \lambda^\alpha$, $p^i_\alpha(e') = \lambda^i_\alpha$, then there exists a solution $s^1\colon U \to E'$ to the H-C equations defined on a neighbourhood of x_0 such that, $y^\alpha(j^1_{x_0}s^1) = \lambda^\alpha$, $p^i_\alpha(j^1_{x_0}s^1) = \lambda^i_\alpha$, $y^\alpha_{,j}(j^1_{x_0}s^1) = \lambda^\alpha_j$, $p^i_{\alpha,j}(j^1_{x_0}s^1) = \lambda^i_{\alpha,j}$.

Applications to General Relativity

Einstein-Hilbert Lagrangian

$p_M\colon M \to N$ bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n$.
(x^i, y_{jk}) coordinate system, where $y_{jk} = y_{kj}$ are defined by,

$$g_x = y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \quad \forall g_x \in (p_M)^{-1}(U).$$

The E-H Lagrangian density is given by

$$(\Lambda_{EH})_{j^2_x g} = g^{ij}(x)(R^g)^h_{ihj}(x)v_g(x) = L_{EH}(j^2_x g)v_x,$$

- v is the standard volume form,
- R^g is the curvature tensor of the Levi-Civita connection Γ^g of g,
- v_g denotes the Riemannian volume form attached to g.

Hence,

$$L_{EH} \circ j^2 g = \rho(y^{ij} \circ g)(R^g)^h_{ihj},$$

where $\rho = \sqrt{(-1)^{n^-} \det((y_{ab})^n_{a,b=1})}$.

In local coordinates:

$$L_{EH} = \rho \left(y^{ac} y^{bd} - y^{ab} y^{cd} \right) y_{ab,cd} + L_0 ,$$

where

$$L_0 = \tfrac{\rho}{2} \sum_{r \leq s} \sum_{k \leq l} F_{kl,i;rs,j} y_{kl,i} y_{rs,j} , \quad F_{kl,i;rs,j} \in C^\infty(M).$$

Hence L_{EH} is an affine function and its P-C form projects onto $J^1 M$.
We have:

$$p^i_{kl} = \sum_{r \leq s} \left(\frac{\partial^2 L_0}{\partial y_{rs,j} \partial y_{kl,i}} - \frac{\partial (L_{EH})^{ij}_{kl}}{\partial y_{rs}} - \frac{\partial (L_{EH})^{ij}_{rs}}{\partial y_{kl}} \right) y_{rs,j} ,$$

$$H = \sum_{k \leq l} \sum_{r \leq s} \left(-\tfrac{1}{2} \frac{\partial^2 L_0}{\partial y_{rs,j} \partial y_{kl,i}} + \frac{\partial (L_{EH})^{ij}_{kl}}{\partial y_{rs}} \right) y_{rs,j} y_{kl,i} .$$

The Hamilton-Cartan equations become

$$\begin{cases} 0 = \dfrac{\partial(p^i_{kl} \circ j^1 s)}{\partial x^i} - \dfrac{\partial H}{\partial y_{kl}} \circ j^1 s, & 1 \leq k \leq l \leq n, \\ 0 = \dfrac{\partial(y_{kl} \circ s)}{\partial x^i} + \dfrac{\partial H}{\partial p^i_{kl}} \circ j^1 s, & 1 \leq i \leq n,\ 1 \leq k \leq l \leq n. \end{cases}$$

Theorem

We have

(i) *The E-H Lagrangian satisfies the regularity condition.*

(ii) *Given symmetric scalars* $\gamma^i_{jk} = \gamma^i_{kj}$, $i, j, k = 1, \ldots, n$, *there exists a Ricci-flat (pseudo-)Riemannian metric of signature* (n^-, n^+) *defined on a neighbourhood of* $x_0 \in N$ *such that,* $g_{ij}(x_0) = \delta_{ij}$, $(\Gamma^g)^i_{jk}(x_0) = \gamma^i_{jk}$, *for all* i, j, k.

R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, 18, Springer-Verlag, New York, 1991.

R. Durán Díaz, J. Muñoz Masqué, *Second-order Lagrangians admitting a second-order Hamilton-Cartan formalism*, J. Phys. A: Math. Gen. **33** (2000), 6003–6016.

Pedro L. García, J. Muñoz Masqué, *Le problème de la régularité dans le calcul des variations du second ordre*, C.R. Acad. Sci. Paris **301** Série I (1985), 639–642.

H. Goldschmidt, S. Sternberg, *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble) **23** (1973), no. 1, 203–267.

J. Muñoz Masqué, *An axiomatic characterization of the Poincaré-Cartan form for second-order variational problems*, Lecture Notes in Math. 1139, Springer-Verlag 1985, pp. 74–84.

D. J. Saunders, M. Crampin, *On the Legendre map in higher-order field theories*, J. Phys. A: Math. Gen. **23** (1990), 3169–3182.